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Essential numerical tools and perturbation analysis (2.a)

Day 2: Solving the Model

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The Linear Model

Now we want to solve the linear model (we drop the hats):

$$E_t\left[Ay_{t+1} + By_t + Cy_{t-1} + De_t\right] = 0$$

where the *recursive* solution is

$$y_t = G_y y_{t-1} + G_e e_t$$

If $y_t \in \mathbf{R}^n$ and $e_t \in \mathbf{R}^{n_e}$ then:

- $$A, B, C, G_y \in \mathbf{R}^n \times \mathbf{R}^n$$
- $$D, G_e \in \mathbf{R}^n \times \mathbf{R}^{n_e}$$

The Problem

Note that if the decision rule satisfies: $y_t = G_y y_{t-1} + G_e e_t$

we have

$$y_{t+1} = G_y y_t + G_e e_{t+1} = G_e e_{t+1} + G_y G_e e_t + G_y G_y y_{t-1}$$

and if we make all substitutions in $E_t\left[Ay_{t+1} + By_t + Cy_{t-1} + De_t\right]$, we get:

$$(AG_yG_y + BG_y + C)y_{t-1} + (AG_yG_e + BG_e + DG_e)e_t = 0$$

This must be true for any y_{t-1} or e_t. This yields the conditions that define G_y and G_e

$$AG_y^2 + BG_y + C$$

$$AG_yG_e + BG_e + D$$

The Riccatti Equation

The transition matix G_e must satisfy a *second order matrix equation*:

$$AX^2 + BX + C$$

From our intuition in dimension 1, we know there must be multiple solutions

- how do we find them?
- how do we select the right ones?

Obviously, the qualitative dynamics of the model are given by $y_t = Xy_{t-1}$

For the solution to the model to be stationary, the spectral radius of X should be smaller than 1.

The State-Space System

It is possible to associate a *linear* system to this Riccatti equation.

It is the *state-space* representation. It characterizes vectors $v_t = (y_t, y_{t+1})$ along any admissible trajectory. These vectors must satisfy:

$$\underbrace{\begin{bmatrix} I & 0 \\ 0 & A \end{bmatrix}}_{F} v_{t+1} = \underbrace{\begin{bmatrix} 0 & I \\ -C & -B \end{bmatrix}}_{G} v_t$$

In particular, we are interested in *fundamental* trajectories, such that $\mu v_{t+1} = \lambda v_t$ where $\mu, \lambda \in \mathbf{R}$.

Warning:

The formulation with a pair of generalized eigenvalues μ, λ is just a technicality meant to avoid infinite eigenvalues in the calculations which can happen when A is defective. To build the intuition, it is suggested to look at the case $\mu = 1$ and $A = I$.

Note that, on a fundamental trajectory, we have $\mu(y_t, y_{t+1}) = \lambda(y_{t-1}, y_t)$.

These trajectories are clearly recursive: $y_t = \frac{\lambda}{\mu} y_{t-1}$

When $\mu = 0$ and $\lambda \neq 0$ we say there is an infinite eigenvalue. Most of the theory works if we forget about μ but consider only $\lambda \in [0, \infty]$

The eigenvalues of the system

According to generalized eigenvalue theory, the system has generically $2n$ fundamental trajectories: $(\mu_1, \lambda_1, v_1), \dots (\mu_{2n}, \lambda_{2n}, v_{2n})$

To simplify our reasoning we can assume that eigenvalues are ranked in increasing eigenvalues (with infinite eigenvalues last):

$$0|\lambda_1| \leq \dots \leq |\lambda_{2n}| \leq \infty$$

Remember that fundamental trajectories are recursive?

It can be shown that any recursive solution X to the quadratic system is obtained, by selecting n different eigenvectors.

As a result, there are exactly $\binom{2n}{n}$ different solutions to our system.

The model is **well defined** when only 1 of all this solutions is non divergent. This is equivalent to say:

$$0 \leq |\lambda_1| \leq \dots \leq \lambda_n \leq 1 < |\lambda_{n+1}| \leq \dots \leq |\lambda_{2n}| \leq \infty$$

Example 1

Forward looking inflation:

$$\pi_t = \alpha \pi_{t+1}$$

with $\alpha > 1$. Is it well defined?

We can rewrite the system as: $\alpha\pi_{t+1} - \pi_t + 0\pi_{t-1} = \pi_{t+1} - (\frac{1}{\alpha} + 0)\pi_t + \left(\frac{1}{\alpha}0\right)\pi_{t-1}$

The eigenvalues are $0 \leq 1 < \frac{1}{\alpha}$. The unique solution is $\pi_t = 0\pi_{t-1}$

Example 2

Debt accumulation equation by a rational agent:

$$b_{t+1} - (1 + \frac{1}{\beta})b_t + \frac{1}{\beta}b_{t-1} = 0$$

Is it well-defined?

The associated polynomial $x^2 - (1 + \frac{1}{\beta})x + \frac{1}{\beta}$ has two eigenvalues $\lambda_1 = 1 < \lambda_2 = \frac{1}{\beta}$

The unique solution is $b_t = b_{t-1}$.

- it is a *unit-root*: any initial deviation in b_{t-1} has persistent effects

Example 3

Productivity process: $z_t = \rho z_{t-1}$ with $\rho < 1$

The generalized eigenvalues are $\lambda_1 = \rho \leq 1 < \lambda_2 = \infty$

More generally, any variable that does not appear in $t+1$ creates one infinite generalized eigenvalue.

Tip

To see where the hidden eigenvalue comes from: make $\lambda \rightarrow \infty$ in the following equation:

$$z_{t+1} - (\lambda + \rho) z_t + \frac{\lambda}{\rho} z_{t-1} = 0$$

Blanchard-Kahn Criterium

Remember the criterium for well-definedness?

$$0 |\lambda_1| \leq \ldots \leq \lambda_n \leq 1 < |\lambda_{n+1}| \leq \ldots \leq |\lambda_{2n}| \leq \infty$$

Now realize (or admit) that for each variable not appearing in $t+1$ in the model, there is an associated infinite eigenvalue.

We can deduce from that a common formulation of the Blanchard-Kahn criterium:

> The model satisfies the Blanchard-Kahn criterium if the number of eigenvalues greater than one, is exactly equal to the number of variables *appearing* in $t+1$.

It is equivalent to the **existence and unicity of a non-divergent recursive solution.**

Computing the solution

There are several classical methods to compute the solution to the algebraic Riccati equation: $AX^2 + BX + C = 0$

- qz decomposition
    - traditionnally used in the DSGE literature
    - a little bit unintuitive but easy to implement from the state-space representation
    - constructive: it produces *all* eigenvalues which makes it easy to check BK conditions

- cyclic reduction
    - more adequate for big models
- linear time iteration
    - very easy to remember/implement

Checking the solution

Cyclic Reduction and Linear Iteration are iterative algorithms that usually converge to a solution X but sometimes fail to do so.

After using one of these algorithms we can check

- that the solution is non divergent:

$$\rho(X)<1$$

- check that the first rejected eigenvalue is smaller than 1:

$$\rho((A X + B)^{-1} A)<1$$

```
md"""## Checking the solution

Cyclic Reduction and Linear Iteration are iterative algorithms that usually converge to a solution $X$ but sometimes fail to do so.

After using one of these algorithms we can check
- that the solution is non divergent:
$$\rho(X)<1$$
- check that the first rejected eigenvalue is smaller than 1:
$$\rho( (A X + B)^{-1} A )<1$$
"""
```

Tip

Using solvant theory, it is possible to show that the eigenvalues of $(A X + B)^{-1} A$ are precisely the inverse of all the eigenvalues that have been rejected while constructing X

```
tip(md"""Using solvant theory, it is possible to show that the eigenvalues of $(A X + B)^{-1} A$ are precisely the inverse of all the eigenvalues that have been rejected while constructing $X$""")
```

Linear Time Iteration (1)

Return to the Ricatti system but suppose that decision rules today and tomorrow are different:

- today: $y_t = \overline{y} + X y_{t-1} + G_y e_t$
- tomorrow: $y_{t+1} = \overline{y} + \tilde{X} y_{t-1} + G_y e_t$

Then the Ricatti equation becomes:

$$A\tilde{X}X + BX + C = 0$$

Linear Time Iteration (2)

The linear time iteration algorithm consists in solving the decision rule X today as a function of decision rule tomorrow $\tilde{X}$. This corresponds to the simple formula:

$$X = -(A\tilde{X} + B)^{-1}C$$

And the full algorithm can be described as:

- choose X_0
- for any X_n, compute $X_{n+1} = T(X_n) = -(AX_n + B)^{-1}C$
  - repeat until convergence

Tip

Linear Time Iteration is a special case of a Bernouilli iteration

Linear Time Iteration (3)

Starting from a random initial guess, the linear time-iteration algorithm usually converges to the solution X with the smallest modulus:

$$\underbrace{|\lambda_1| \leq \cdots \leq |\lambda_n|}_{\text{Selected eigenvalues}} \leq |\lambda_{n+1}| \cdots \leq |\lambda_{2n}|$$

In other words, it finds the right solution when the model is well specified.

Then you just need to check that first rejected eigenvalue is greater than 1.

> **Warning:**
>
> In some cases, there is no convergence. For instance if $|\lambda_n| = |\lambda_{n+1}|$). Or for a specific initial value X_0 such that some $AX_n + B$ is not invertible. However when the algorithm converges, it always satisfies the above condition.

Exercise

Finish the solution of the RBC model.

Copy and paste the code for the model from session 1.

```
md"""__Copy and paste the code for the model from session 1.__"""
```

```
Enter cell code...
```

Use ForwardDiff to compute A,B,C,D

```
md"""__Use ForwardDiff to compute A,B,C,D__"""
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Enter cell code...
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Implement the time-iteration algorithm to solve for G_y

```
md"""__Implement the time-iteration algorithm to solve for $G_y$__"""
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Enter cell code...
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Check that the solution solves the original problem

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md"__Check that the solution solves the original problem__"
```

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Enter cell code...
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Check that the greatest eigenvalue of the solution is smaller than 1

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md"__Check that the greatest eigenvalue of the solution is smaller than 1__"
```

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Enter cell code...
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Check that the first excluded eigenvalue is greater than 1.

```
md"__Check that the first excluded eigenvalue is greater than 1.__"
```

Compute G_e

```
md"__Compute $G_e$__"
```

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Enter cell code...
```

Bonus: compute the generalized eigenvalues of state-space system. Are they consistent with what you have found?

```
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consistent with what you have found?__"
```

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Enter cell code...
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Bonus: plot some impulse response functions.

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md"""__Bonus: plot some impulse response functions.__"""
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