Dynamic Portfolios in DSGE Models

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Abstract

This paper uses the bifurcation theory to characterize dynamic portfolio choices in a DSGE model. It provides theoretical support for the recent methodology developed in Devereux and Sutherland [2009a] and Tille et al. [2010]. While the original method is restricted to general equilibrium models with two or more agents, the bifurcation approach itself can be applied to partial equilibrium settings where interest rates are exogenous. I illustrate this by approximating the strategic asset allocation for a long term investor with intermediate consumption. I also assess the accuracy of the bifurcation method by comparing its result with a more precise global solution method. If find that bifurcation portfolios approximate the true portfolios accurately in a two-countries settings with symmetric preferences, where the actual portfolio decision rules are close to linear.

1 Introduction

The increased financial integration of the last two decades has triggered a growing interest in knowing the behavior of international financial investors. However, until recently, it was not possible to characterize optimal portfolios in a standard international macroeconomic model with many assets.

In a static context, Judd and Guu [2001] have provided the intuition that optimal portfolio choice can be characterized asymptotically when shocks are small enough by resorting to a bifurcation theorem. In particular, this approach highlights the existence of a special bifurcation point which is the limit of optimal portfolios when shocks tend to zero.

In two popular methodological paper, Devereux and Sutherland [2008, 2009a] have provided efficient formulas to compute these bifurcation portfolios in a dynamic stochastic general equilibrium without relying explicitly on the bifurcation theory. Simultaneously, Tille et al. [2010] have proposed an iterative procedure yielding the same bifurcation portfolios. In contrast to Judd and Guu [2001], their computations, are restricted to general equilibrium models, for which all returns are equal in the deterministic equilibrium.

In this paper, I adapt the bifurcation argument from Judd and Guu [2001] to dynamic portfolio problems. This gives a formal justification for results in

Devereux and Sutherland [2008, 2009a] as well as a consistent framework to extend it. It also provides simple conditions at which the iterative method as proposed by Tille et al. [2010] to solve for the portfolios will yield valid results. I show that the same approach can be used to characterize optimal portfolios in a partial equilibrium setup in which at least one financial return is exogenous. This last feature makes it possible to solve for optimal external asset allocation by a small open economy. I believe it can complement the international portfolio literature (reviewed by Pavlova [2010]) by providing a way to check whether observed anomalies (small risk-premium, home bias) conflict with general equilibrium assumptions or with standard specifications of utility functions.

Accuracy checks show indeed that portfolio decisions have more curvature with respect to the level of wealth in a partial equilibrium setup than in a general equilibrium where price changes can hinder asymmetric alteration of tastes. As a result I conclude that the bifurcation approach that has been used in the international portfolio literature is indeed quite precise, at least with endowment economies. In that case, the true decision rule is indeed quasi-linear.

Other solution methods have been developed to solve for portfolios in DSGE models. The former attempts have relied on the assumption of locally complete markets (e.g. Coeurdacier and Gourinchas [2008], Coeurdacier et al. [2010]), or ad-hoc iterative methods (Heathcote and Perri [2007]). In another strand of the literature, Campbell and Viceira has developed a numerical method to determine strategic asset allocation with intermediate consumption of long-term investors. This approach has been adapted to general equilibrium and adapted in a two-country model (Evans et al. [2008], Hnatkovska [2010]). While these authors rely on approximations coming from continuous time finance and use custom iterations on price and quantities, we formulate our solution in a generic DSGE framework that has proven its flexibility.

Section 2 exposes the problem of portfolio indeterminacy using a formalization that is common to partial and general equilibrium models. It also introduces the concept of a bifurcation portfolio. Section 3 shows how to recover a a Taylor approximation of this portfolio using well known methods to solve for regular DSGE models. In particular it shows how to adapt the latter to tackle the partial equilibrium case. Section 4 provides a simple strategic asset allocation example and section 5 shows a classic application on a two-countries endowment model. Both sections 4 and 5 feature accuracy checks and a comparison with a global solution method. Section **??** concludes.

2 Dynamic portfolio problems

2.1 First order degeneracy

s the state-space $\mathcal{S}$ be a closed convex subset of R^s. Let a functional space $\mathcal{D}^n$ denote the set of decision rules over $\mathcal{S}$ with values in R^n. $\mathcal{D}^n$ is a subset of all the functions $\mathcal{S} \rightarrow R^n$ with a norm $\|.\|$. I assume that $(\mathcal{D}^n, \|.\|)$ is a Banach space.

Given a transition function g, let us define a controlled process with values in $\mathcal{S}$ by

$$\begin{aligned} s_t &= g\left(s_{t-1}, x_{t-1}, p_{t-1}, \lambda\epsilon_t\right) \\ s_0 &= \bar{s} \end{aligned} \tag{2.1}$$

where x_{t-1},p_{t-1} are two controls taken as a function of the state s_{t-1}. $\bar{s} \in \mathcal{S}$ the initial state. The exogenous process (ϵ_t) is a normal i.i.d shock, scaled by a parameter $\lambda \in [0, 1]$.

Solving the model with the level of risk λ consists in finding the optimal decision rules $\varphi \in \mathcal{D}^x$ for regular variables x_{t-1} and $\psi \in \mathcal{D}^p$ for portfolio variables p_{t-1} such that by setting at all dates t

$$\begin{aligned} x_t &= \varphi\left(s_t\right) \\ p_t &= \psi\left(s_t\right) \end{aligned} \tag{2.2}$$

the following criteria are met by the resulting process:

$$\begin{aligned} 0 &= E_t f\left(s_t, x_t, p_t, s_{t+1}, x_{t+1}\right) \\ 0 &= E_t h\left(s_t, x_t, p_t, s_{t+1}, x_{t+1}\right) \end{aligned} \tag{2.3}$$

Those criteria are typically Euler's first order conditions which are derived by the optimization of an inter-temporal objective. I assume that f and h are smooth functions with values in $R^x$and R^p respectively.

Using these optimality conditions, one can rewrite the problem as a pure functional equation. Let $F(\varphi, \psi, \lambda)$ and $H(\varphi, \psi, \lambda)$ denote two functions defined over S such that:

$$\begin{aligned} F(\varphi, \psi, \lambda)(s) &= E_t f\left(s, \varphi(s), \psi(s), g\left(s, \varphi(s), \psi(s), \lambda\epsilon\right), \varphi\left(g\left(s, \varphi(s), \psi(s), \lambda\epsilon\right)\right)\right) \\ H(\varphi, \psi, \lambda)(s) &= E_t h\left(s, \varphi(s), \psi(s), g\left(s, \varphi(s), \psi(s), \lambda\epsilon\right), \varphi\left(g\left(s, \varphi(s), \psi(s), \lambda\epsilon\right)\right)\right) \end{aligned}$$

I assume that these two functions are part of the Banach spaces $\mathcal{D}^x$ and $\mathcal{D}^p$ respectively. Then solving the original system is equivalent to find the solution φ and ψ of the functional system:

$$\begin{aligned} F(\varphi, \psi, \lambda) &= 0 \\ H(\varphi, \psi, \lambda) &= 0 \end{aligned}$$

Until now I have not specified what makes the portfolio variables specific. Let's make the two following assumptions:

- For any $\psi \in \mathcal{D}^p$ and for any $\lambda \in [0,1]$ the system

$$\begin{aligned} s_t &= g\left(s_{t-1}, x_{t-1}, \psi\left(s_{t-1}\right), \lambda \epsilon_t\right) \\ 0 &= E_t f\left(s_t, x_t, \psi\left(s_t\right), s_{t+1}, x_{t+1}\right) \end{aligned}$$

  is a regular DSGE problem which has one unique solution $\varphi(\psi, \lambda)$.

- For any $\psi \in \mathcal{D}^p$

$$H\left(\varphi\left(\psi, 0\right), \psi, 0\right) = 0$$

The first condition associates the choice of the portfolios to the specific portfolio equation. For any portfolio choice, the rest of the model can be solved in a regular way. The second condition states that portfolio choice are indeterminate in a perfect foresight version of the model. Any portfolio choice can lead to an admissible perfect foresight solution.

2.2 The bifurcation portfolio

We have seen in the last section that the model does not have a unique solution at $\lambda = 0$. The bifurcation approach consists in looking at the limit portfolio when λ tends towards 0. To apply the bifurcation theory rigorously, we need to assume that the application $(\psi, \lambda) \rightarrow H\left(\varphi\left(\psi, \lambda\right), \varphi, \lambda\right)$ is smooth enough. We will also use the Banach structure of the sets $\mathcal{D}^n$ to which ψ belongs.

The general approach consists in computing derivatives of the portfolio criterion with respect to λ until a non-zero derivative is found. Using the fact that ϵ has a zero mean, we always have:

$$\frac{\partial}{\partial \lambda} H\left(\varphi\left(\psi, 0\right), \psi, 0\right)$$

Going to the second order will in general be enough to pin down ψ as it will include second-order expectations of the shocks. I will call ψ_0 the bifurcation portfolio if it satisfies the three following conditions:

- $$\frac{\partial^2}{\partial \lambda^2} H\left(\varphi\left(\psi_0, 0\right), \psi, 0\right) = 0$$

- The linear operator $\psi \mapsto \frac{\partial}{\partial \psi} \frac{\partial^2}{\partial \lambda^2} H\left(\varphi\left(\psi_0, 0\right), \psi, 0\right)$ has an inverse U

- The linear operator $\psi \mapsto \frac{\partial}{\partial \psi} \frac{\partial^2}{\partial \lambda^2} H\left(\varphi\left(\psi_0, 0\right), \psi, 0\right)$ is bounded

The first condition states that the bifurcation satisfies the optimality condition up the second order in λ.

The second condition basically implies that the function $\psi \rightarrow \frac{\partial^2}{\partial \lambda^2} H\left(\varphi\left(\psi, 0\right), \psi, 0\right) = 0$ defines ψ_0 uniquely. The third assumption is a regularity assumption which is necessary when applying the bifurcation theorem to infinitely dimensional Banach spaces.

These three conditions are precisely those of the bifurcation theorem **??** given in the appendix applied to $\Phi(\psi,\lambda) = H(\varphi(\psi,\lambda),\psi,\lambda)$.

The second part of the bifurcation theorem also states how portfolios at $\lambda > 0$ can be recovered from this bifurcation portfolio. We have:

$$\begin{aligned}
\psi_\lambda &= \psi_0 + \psi_1\lambda + \psi_2\frac{\lambda^2}{2} + o\left(\lambda^2\right) \\
\psi_1 &= -\frac{1}{2}U.\left(\Phi^{'''}_{\sigma\sigma\sigma}(\psi_0,0)\right) \\
\psi_2 &= -\frac{1}{3}U.\left(3\Phi^{'''}_{\alpha\sigma\sigma\sigma}\psi_0^{'} + 2\Phi^{'''}_{\alpha\alpha\sigma\sigma}.[\psi_1,\psi_1] + \Phi^{''''}_{\sigma\sigma\sigma\sigma}\right)
\end{aligned}$$

In this expression, the error term $o(\lambda)$ represents a function of the Banach space $\mathcal{D}^p$ whose norm is negligible w.r.t. λ^2.

Given the fact that third order moments of ϵ_t are zero, these equations simplify to:

$$\begin{aligned}
\psi_1 &= 0 \\
\psi_2 &= -\frac{1}{3}U.\Phi^{''''}_{\sigma\sigma\sigma\sigma}
\end{aligned}$$

These simple computations show that the bifurcation portfolio is in general accurate up to first order in λ.

3 Taylor expansion of the bifurcation portfolio

In this section, I describe how perturbation methods can be used to compute a Taylor expansion of the bifurcation portfolio rule. I assume that decision functions in D^n have enough regularity to be represented as a Taylor expansion around the steady-state $\bar{s}$ of the model. In particular, I assume that the regular controls satisfy:

$$\begin{aligned}
x(s) &= \bar{x} + x_1[s-\bar{s}] + \frac{1}{2}x_2[s-\bar{s}]^2 + ... + \frac{1}{n!}x_n[s-\bar{s}]^n + o\left([s-\bar{s}]^n\right) \\
p(s) &= \bar{p} + p_1[s-\bar{s}] + \frac{1}{2}p_2[s-\bar{s}]^2 + ... + \frac{1}{n!}p_n[s-\bar{s}]^n + o\left([s-\bar{s}]^n\right)
\end{aligned}$$

Also, the optimality criterion $\Phi(\psi,\lambda)$ which belongs to $\mathcal{D}^p$ can be extended as:

$$
\begin{aligned}
H\left(\varphi\left(\psi,\lambda\right),\varphi,\lambda\right) &= \bar{H} + H_1\left[s-\bar{s}\right] + \frac{1}{2}H_2\left[s-\bar{s}\right]^2 + ... + \frac{1}{n!}H_n\left[s-\bar{s}\right]^n + ... \\
&+\lambda\left[\bar{H}_\lambda + H_{1\lambda}\left[s-\bar{s}\right] + \frac{1}{2}H_{1\lambda}\left[s-\bar{s}\right]^2 + ... + \frac{1}{n!}H_{n\lambda}\left[s-\bar{s}\right]^n + ...\right] \\
&+\frac{\lambda^2}{2}\left[\bar{H}_{\lambda\lambda} + H_{1\lambda\lambda}\left[s-\bar{s}\right] + \frac{1}{2}H_{2\lambda\lambda}\left[s-\bar{s}\right]^2 + ... + \frac{1}{n!}H_{n\lambda\lambda}\left[s-\bar{s}\right]^n + ...\right] \\
&+... \qquad (3.1)
\end{aligned}
$$

where:

$$
\frac{\partial^2}{\partial\lambda^2}H\left(s,\lambda\right) = \bar{H}_{0\lambda\lambda} + H_{1\lambda\lambda}\left[s-\bar{s}\right] + \frac{1}{2}H_{2\lambda\lambda}\left[s-\bar{s}\right]^2 + ... + \frac{1}{n!}H_{n\lambda\lambda}\left[s-\bar{s}\right]^n + o\left(\left[s-\bar{s}\right]^n\right)
$$

For any given coefficients of the portfolio rule (p_k), we can compute in turn the solution to the regular DSGE problem to get the coefficients (x_k) and use these coefficients together to compute the coefficients $(H_{k\lambda\lambda})$ of the optimality criterion. This can be done easily provided that we know how to compute a higher order approximation of a regular DSGE model. Such a procedure is described in Fackler [2005], Jin and Judd [2002].

The coefficients of the bifurcation portfolio must be such such that $\frac{\partial^2}{\partial\lambda^2}H = 0$ implying that all coefficients $\left(H_{k\lambda\lambda}\right)_k$ are zero. . If for any k, $H_{k\lambda\lambda}$ only depends on the coefficients p_l with $l \leq k$, a simple recursive procedure can be used to solve for the coefficients: numerically solve for p_0 such that $H_{0\lambda\lambda} = 0$, then for each i taking as given the coefficients $(p_0, ..., p_{i-1})$ find p_i such that $H_{i\lambda\lambda} = 0$. Each of the optimal p_i can be numerically computed by applying (for instance) a newton optimization procedure to the function $p_i \to H_{i\lambda\lambda}$.

For each p_i to be well defined, the matrix M_{iij} representing the application $p_i \to H_{i\lambda\lambda}$ must be invertible when evaluated at the solution. In fact, this is a manifestation of a more general condition. Let's represent the derivative $\frac{\partial H_{\lambda\lambda}}{\partial p}$ of the application $p \to H_{\lambda\lambda}$ by an infinite matrix $M = (M_{ij})$ where each submatrix M_{ij} represents the derivative $\frac{\partial H_{i\lambda\lambda}}{\partial p_j}$. For the bifurcation portfolio to be well defined, the derivative $\frac{\partial H_{\lambda\lambda}}{\partial p}$ needs to be invertible and as a result (M_{ij}) must be invertible.

Under the conditions stated in the preceding paragraph because the matrix (M_{ij}) is lower tridiagonal (see following table). Hence it is invertible if and only if all its diagonal blocks are invertible.

p

		p_0	p_1	p_2	p_3	...
	$H_{0\lambda\lambda}$	M_{00}	0	0	0	$\cdots$
	$H_{1\lambda\lambda}$	M_{10}	M_{11}	0	0	$\cdots$
$H_{\lambda\lambda}$	$H_{2\lambda\lambda}$	M_{20}	M_{21}	M_{22}	0	$\cdots$
	$H_{3\lambda\lambda}$	M_{30}	M_{31}	M_{32}	M_{33}	$\cdots$
	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Each coefficient of (M_{ij}) can be computed numerically (and is likely to have been computed when looking for p_i). Checking that this matrix is invertible gives an indication regarding the applicability of the bifurcation theory. If it is not invertible, the bifurcation portfolio is not uniquely defined.

In theory, the inverse of this matrix needs to be actually computed in order to recover higher order coefficients of the portfolio decision rule. However, the result of the preceding section show that it is not necessary unless one is willing to approximate second order coefficients of the portfolio rule (w.r.t. λ). Given that it implies a fourth order approximation of the optimality criterion (w.r.t. λ) and even higher order expansions of the non portfolios decision rules (w.r.t $[s-\bar{s}]$) it is not clear whether it is of any practical use: one may be happy enough with the bifurcation portfolio.

3.1 Dealing with non zero expected excess returns

Until now I have assumed that under perfect foresight, the portfolio equations were satisfied for any portfolio decision rule. We can also encounter the less favorable case in which some of them are never met in the perfect foresight model. In that case, it is possible to replace the transition equations by alternative ones which are identical at $\lambda = 1$ and more regular at $\lambda = 0$.

Let's replace the model (2.1,2.2,2.3) by a set of equations:

$$\begin{aligned} s_t &= g\left(s_{t-1}, x_{t-1}, p_{t-1}, \lambda\epsilon_t, \lambda^2\right) \\ 0 &= E_t f\left(s_t, x_t, s_{t+1}, x_{t+1}\right) \\ 0 &= E_t h\left(s_t, x_t, s_{t+1}, x_{t+1}\right) \end{aligned} \tag{3.2}$$

with the law of motion for states variables being allowed to depend on the level of risk. We can now follow the exact same steps as before if we define the optimality criterion as:

$$H(\varphi, \psi, \lambda)(s) = E_t h\left(s, \varphi(s), g\left(s, \varphi(s), \psi(s), \lambda\epsilon, \lambda^2\right), \varphi\left(g\left(s, \varphi(s), \psi(s), \lambda\epsilon, \lambda^2\right)\right)\right)$$

Except for these two slight changes, the procedure is exactly the same. In particular, the conditions are the same for the bifurcation to be well defined.

3.2 Practical implementation

To solve for first order portfolios, it is convenient to augment the original model with additional control variables z_t:

$$\begin{aligned} s_t &= g\left(s_{t-1}, x_{t-1}, p_{t-1}, \lambda\epsilon_t, \lambda^2\right) \\ 0 &= E_t f\left(s_t, x_t, s_{t+1}, x_{t+1}\right) \\ z_t &= E_t h\left(s_t, x_t, s_{t+1}, x_{t+1}\right) \\ p_t &= P_0 + P_1 s_t \end{aligned}$$

where unknown coefficient matrices $\bar{p}$ and P_1 characterize our unknown first order decision rule.

Using any guess for P_0, P_1 and using the steps given in appendix **??**, we get directly a third order approximation of all the controls, including z_t. The Taylor expansion of z_t corresponds exactly 3.1. It contains terms in $Z_{0\lambda\lambda}, Z_{1\lambda\lambda}$. All we need is to numerically look for P_0 and P_1 such that $Z_{0\lambda\lambda} = 0$ and $Z_{1\lambda\lambda} = 0$.

4 Example 1: strategic asset allocation

I apply the method from section 3 to a variant of the model described in Campbell et al. [2001]: optimal allocation from infinitely-lived investor with Epstein-Zin recursive preferences were asset returns are assumed to follow a VAR(1). I augment it with an exogenous non portfolio income. Because of this addition, the method described in Campbell et al. [2001] is not applicable because it relies on the existence of a closed form solution when inter-temporal elasticity of substitution is equal to 1.

Let's consider an inter-temporal problem for an agent with a stochastic exogenous income (y_t) who chooses between either consuming or saving it. The consumed part is denoted by (c_t). I assume that a fraction (ξ_t) of the savings is remunerated at exogenous rate $\left(r_t^1\right)$ while the remaining part $(1-\xi_t)$ is remunerated at $\left(r_t^2\right)$. Let W_t denote available income at the beginning of period t. Its law of motion is given by

$$W_t = y_t + (W_{t-1} - c_{t-1})\left(\xi_{t-1} r_t^1 + (1-\xi_{t-1})\, r_t^2\right)$$

where the consumption decision c_{t-1} and the portfolio choice ξ_{t-1} are the controls taken as a function of W_{t-1}. I assume that both exogenous income and financial returns are independent i.i.d. normal shocks:

$$\begin{aligned} y_t &= \bar{y} + \epsilon_t^y \\ r_t^1 &= \bar{r}^1 + \epsilon_t^1 \\ r_t^2 &= \bar{r}^2 + \epsilon_t^2 \end{aligned}$$

Let's define overall returns on the portfolio by:

$$r_t = \xi_{t-1} r_t^1 + (1-\xi_{t-1})\, r_t^2$$

Because of the absence of correlation, available income W_t is the sole state variable.

The optimal choice for $c(.)$ and $\xi(.)$ maximizes utility over the stochastic consumption stream (c_t). This utility is defined recursively as an Epstein-Zin utility function:

$$U_t = U\left(c_t, E_t\left(U_{t+1}\right)\right) = \left[(1-\delta)\, c_t^{\frac{1-\gamma}{\theta}} + \delta\left(E_t\left(U_{t+1}^{1-\gamma}\right)\right)^{\frac{1}{\theta}}\right]^{\frac{\theta}{1-\gamma}}$$

where $\delta \in]0,1[$ is the time-discount factor, γ the relative risk aversion and $\theta = \frac{1-\gamma}{\left(1-\frac{1}{\psi}\right)}$ is a parameter defined depending on the elasticity of inter-temporal substitution ψ. It nests the case of a separable power utility when $\gamma = \frac{1}{\psi}$ and of log utility of $\gamma = \frac{1}{\psi} = 1$.

Euler equations associated to optimal choice are written:

$$
\begin{aligned}
E_t \left[\left\{ \delta \left(\frac{c_{t+1}}{c_t} \right)^{-\frac{1}{\gamma}} \right\}^{\theta} \left(\xi_t r^1_{t+1} + (1-\xi_t)\, r^2_{t+1} \right)^{-(1-\theta)} r^1_{t+1} \right] &= 1 \\
E_t \left[\left\{ \delta \left(\frac{c_{t+1}}{c_t} \right)^{-\frac{1}{\gamma}} \right\}^{\theta} \left(\xi_t r^1_{t+1} + (1-\xi_t)\, r^2_{t+1} \right)^{-(1-\theta)} r^2_{t+1} \right] &= 1
\end{aligned}
$$

A linear combination of the last two equations allows us to distinguish between the optimal inter-temporal savings (equation 4.1 and the portfolio composition (equation 4.2).[1]

$$
E_t \left[\left\{ \delta \left(\frac{c_{t+1}}{c_t} \right)^{-\frac{1}{\gamma}} \right\}^{\theta} \left(\xi_t r^1_{t+1} + (1-\xi_t)\, r^2_{t+1} \right)^{\theta} \right] = 1 \tag{4.1}
$$

$$
E_t \left[\left\{ \delta \left(\frac{c_{t+1}}{c_t} \right)^{-\frac{1}{\gamma}} \right\}^{\theta} \left(\xi_t r^1_{t+1} + (1-\xi_t)\, r^2_{t+1} \right)^{-(1-\theta)} \left(r^2_{t+1} - r^1_{t+1} \right) \right] = 0 \tag{4.2}
$$

The last one is the portfolio equation of the model.

4.1 Solving the model

4.1.1 Portfolio indeterminacy

Under perfect foresight the Euler equations of the model can be replaced by :

$$
\begin{aligned}
\delta \left(\frac{c_{t+1}}{c_t} \right)^{-\frac{1}{\gamma}} \left(\xi_t r^1_{t+1} + (1-\xi_t)\, r^2_{t+1} \right) &= 1 \\
r^1_{t+1} &= r^2_{t+1}
\end{aligned}
$$

According to the second equation, under perfect foresight the two assets must yield the same return in all states of the world. When this is true the overall return on the portfolio $\left(\xi_t r^1_{t+1} + (1-\xi_t)\, r^2_{t+1} \right)$ doesn't depend on ξ_t at all and any portfolio choice is consistent with an optimal perfect foresight path. This assumption is needed in order to apply the bifurcation theorem.

[1] Note, that this manipulation is made for explanatory purpose only. Keeping the original Euler equation for the second asset as the portfolio equation would lead to the same exact result.

Since our two exogenous assets don't meet this requisite the solution consists in replacing the initial problem with a continuum of problems indexed by $\lambda \in [0,1]$ where the two returns are defined as:

$$\begin{aligned} r_t^1 \quad (\lambda) &= \quad \bar{r}^1 + \lambda \epsilon_t^1 \\ r_t^2 \quad (\lambda) &= \quad \bar{r}^1 + \lambda^2 \left(\bar{r}^2 - \bar{r}^1 \right) + \lambda \epsilon_t^2 \end{aligned}$$

With this new definition, the assets are perfect substitute under perfect foresight ($\lambda = 0$) and are unchanged when volatility is taken into account ($\lambda = 1$).

4.1.2 Solution of the non portfolio equations

Given a Taylor expansion of the portfolio choice:

$$\xi_t = \xi_0 + \xi_1 \left(W_t - \bar{W} \right) + ...$$

we can solve the non portfolio equation (4.1) to get a Taylor expansion of the consumption choice (including the dependence on λ):

$$\begin{aligned} c_t \quad &= \quad c_0 + c_1 \left(W_t - \bar{W} \right) + ... \\ &+ \quad \frac{\lambda^2}{2} \left(c_{\lambda\lambda} + c_{1\lambda\lambda} \left(W_t - \bar{W} \right) + ... \right) \\ &+ \quad ... \end{aligned}$$

By construction, the coefficients $c_0, c_1, \cdots$ don't depend on the coefficients of ξ, a manifestation that assets are perfectly substitute.[2]

4.1.3 Optimality criterion

Using the decision rules $\xi(W_t)$ and $c(W_t, \lambda)$ we consider the optimality criterion:

$$H(W_t, \lambda) = E_t \left[(c_{t+1})^{-\frac{\theta}{\gamma}} \left(\xi_t r_{t+1}^1 + (1 - \xi_t) r_{t+1}^2 \right)^{-(1-\theta)} \left(r_{t+1}^2 - r_{t+1}^1 \right) \right]$$

The bifurcation portfolio is characterized by the condition $H''_{\lambda\lambda}(W_t, 0) = 0$. As described in 3.2, we can compute numerically the corresponding Taylor expansion:

$$H''_{\lambda\lambda}\left(\bar{W}, 0 \right) = H''_{0\lambda\lambda} + H'''_{1\lambda\lambda} \left(W_t - \bar{W} \right) + \cdots$$

To find the coefficients (ξ_i) such that all $(H_{i\lambda\lambda})$ are all 0, we need to ensure that each coefficient $H_{i\lambda\lambda}$ doesn't depend on $(\xi_k)_{k>i}$. In general it can be

[2] By following the computation procedure described in the appendix we also get that $c_{i\lambda\lambda}$ doesn't depend on ξ_j for $j > i$.

checked numerically (see next section) but in the current example, we can show it formally. Let's rewrite:

$$H(W_t, \lambda) = E_t \left[A(W_t, \lambda) B(W_t, \lambda) C(W_t, \lambda) \right]$$

where:

$$\begin{aligned}
A(W_t, \lambda) &= \left(\xi_t r_{t+1}^1 + (1 - \xi_t) r_{t+1}^2 \right)^{-(1-\theta)} \\
B(W_t, \lambda) &= \left(\xi_t r_{t+1}^1 + (1 - \xi_t) r_{t+1}^2 \right)^{-(1-\theta)} \\
C(W_t, \lambda) &= \left(r_{t+1}^2 - r_{t+1}^1 \right)
\end{aligned}$$

Given our reformulation of asset returns, we have:

$$\begin{aligned}
C(W_t, 0, 0) &= 0 \\
C'_\lambda(W_t, 0) &= \epsilon^2 - \epsilon^1 \\
C''_\lambda(W_t, 0) &= \bar{r}^2 - \bar{r}^1
\end{aligned}$$

Given that $C(W_t, 0, 0) = 0$ we need to differentiate the two other factors up to order 1 only. We get:

$$\begin{aligned}
B(W_t, 0) &= \bar{r}^\theta \\
B'_\lambda(W_t, 0) &= \theta \bar{r}^{\theta-1} (1 - \xi_t) \left(\epsilon^2 - \epsilon^1 \right)
\end{aligned}$$

The derivation of $A(W_t, \lambda)$ depends on the decision rule for present and future consumption. Using the wealth accumulation equation and making the scale factor apparent:

$$\begin{aligned}
c_{t+1} &= c(W_{t+1}, \lambda) \\
&= c\left(y_{t+1} + (W_t - c_t) \left(\xi_t r_{t+1}^1 + (1 - \xi_t) r_{t+1}^2 \right), \lambda \right) \\
&= c\left(\bar{y} + \lambda \epsilon_{t+1}^y + (W_t - c(W_t, \lambda)) \left(\bar{r} + (1 - \xi_t) \left(\lambda \left(\epsilon^2 - \epsilon^1 \right) + \frac{\lambda^2}{2} \left(\bar{r}^2 - \bar{r}^1 \right) \right) \right), \lambda \right)
\end{aligned}$$

It follows:

$$\begin{aligned}
c_{t+1}\big|_{\lambda=0} &= c\left(\bar{y} + (W_t - c(W_t, 0)), 0 \right) \\
\left. \frac{\partial c_{t+1}}{\partial \lambda} \right|_{\lambda=0} &= c'_W \left(\bar{y} + (W_t - c(W_t, 0)), 0 \right) \left[\epsilon_{t+1}^y + (W_t - c(W_t, 0)) (1 - \xi_t) \left(\epsilon_{t+1}^2 - \epsilon_{t+1}^1 \right) \right]
\end{aligned}$$

And:

δ	0.96
γ	40
$\bar{y}$	1
σ^y	0.005
r^1	$\frac{1}{\beta}$
σ^1	0.001
r^2	$\frac{1}{\beta}+0.05$
σ^2	0.008

Table 1: Calibration

	constant	$W_t - \bar{W}$	$\left(W_t - \bar{W}\right)^2$
c_t	1.3936	0.0400	-0.0000
ξ_t	0.7666	-0.0723	$-$

Table 2: Coefficients of the decision rule at $\bar{W} = 8$

$$
\begin{aligned}
A\left(W_t,0\right) &= \left(c_{t+1}|_{\lambda=0}\right)^{\theta} \\
A'_{\lambda}\left(W_t,0\right) &= \theta\left(c_{t+1}|_{\lambda=0}\right)^{\theta-1}\left(\left.\frac{\partial c_{t+1}}{\partial\lambda}\right|_{\lambda=0}\right)
\end{aligned}
$$

Finally, the second order expansion of the portfolio optimality condition is:

$$
\begin{aligned}
H''_{\lambda\lambda}\left(W_t,0\right) &= E_t\left[A\left(W_t,0\right)B\left(W_t,0\right)C''_{\lambda\lambda}\left(W_t,0\right)\right] \\
&+ E_t\left[A\left(W_t,0\right)B'_{\lambda}\left(W_t,0\right)C'_{\lambda}\left(W_t,0\right)\right] \\
&+ E_t\left[A'_{\lambda}\left(W_t,0\right)B\left(W_t,0\right)C'_{\lambda}\left(W_t,0\right)\right]
\end{aligned}
$$

According to the preceding computations, the function $H''_{\lambda\lambda}$ depends on the decision rules $c\left(W_t,0\right)$, $c'_W\left(W_t,0\right)$ and $\xi\left(W_t\right)$. It follows directly that $H_{k\lambda\lambda} = \frac{\partial^k H''_{\lambda\lambda}}{\partial(W_t)^k}$ depends on the derivatives $\xi^{(i)}\left(W_t\right)$ for $i \leq k$ (and on the derivatives of $c\left(W_t,0\right)$ and $c'_W\left(W_t,0\right)$ that are independent of $\xi\left(\right)$).

This proves that the Jacobian matrix of the portfolio criterion is lower diagonal, which is a necessary condition to apply the bifurcation theorem.

4.2 Calibration and accuracy tests

The mean $\bar{y}$ of exogenous endowment is normalized to 1. The volatility of (y_t) is 0.005 corresponding to a 7% deviation. The "risk-free" and the risky asset have volatilities of 0.001 and 0.008 corresponding to 3.2% and 8.9% standard deviations respectively.

I assume the mean return of the risk-free asset to be the inverse of the time discount factor δ and choose a 5% risk premium between the two assets. I

	ξ_0	ξ'_W
H_0	-0.0093	0.0000
H'_W	-0.0001	-0.0093

Table 3: Jacobian of the bifurcation problem

I compute the bifurcation portfolios following the steps outlined in 3 choosing an approximation point $\bar{W}$ such that the agent is a net saver, i.e. $\bar{W} - \bar{c} > 0$. The Jacobian matrix of the optimality criterion is shown in table 3. In line with the discussion in subsection 4.1.3 we verify that it is lower-triangular indicating that the problem can be well defined.

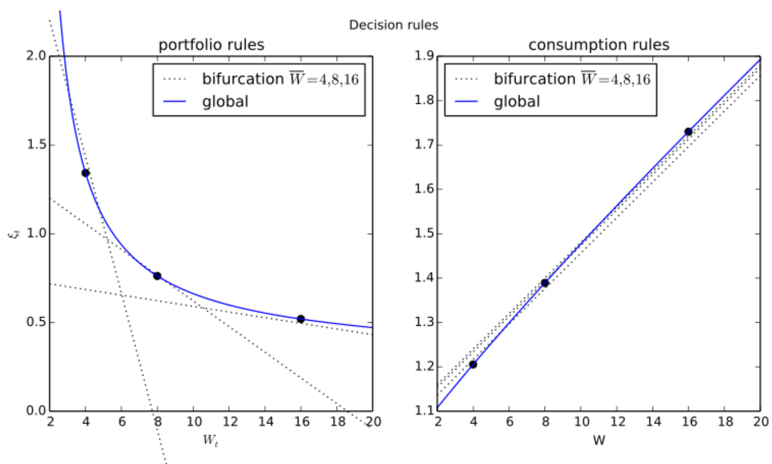

Figure 4.1: Decision rules
The graphs compare first order Taylor expansions of the second order decision rules (dotted line) produced by the bifurcation procedure, computed with a globally approximated decision rule using splines (plain blue).

The dotted curves in figure 4.1 represent the bifurcation portfolios obtained using level of wealth $\bar{W} = 4, 8, 16$ as approximation points. I contrast them with the true solution, represented by the blue line and obtained using a standard global approximation methods. . Left and right panels of figure 4.1 show the decision rules for the portfolio and the consumption respectively. Note that the weight on the risky asset decreases when the wealth increases. This is a normal implication of the utility function that features constant relative risk aversion over static gambles. If financial assets were the sole source of wealth, the share of risk-free and risky assets would remain constant. But given that part of the wealth consists in non-financial income, it follows that the share of risky assets among financial assets must be a decreasing function of wealth, and consequently a decreasing function of the portfolio size.

A direct inspection of these decision rules also produces some insights about

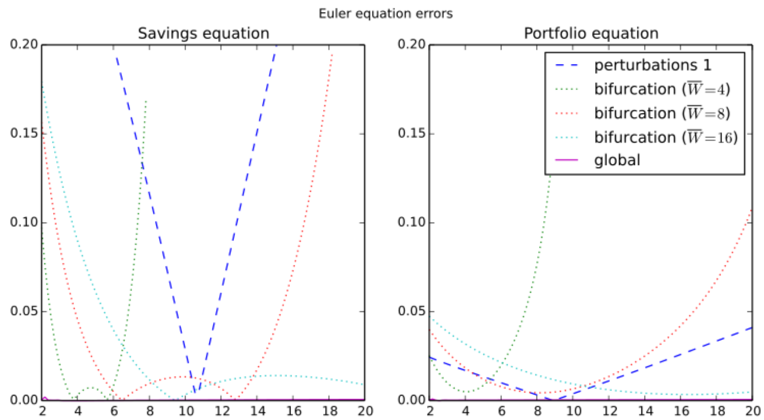

Figure 4.2: Euler equations errors

The graphs show the error made on the Euler equations, at each level of wealth for various approximation methods. It is apparent here that errors made by the global approximation, are negligible with respect to the Taylor expansions.

the accuracy of our approximation. First, we see that the slope of the bifurcation portfolio mimics the slope of the true rule at the approximation points fairly well. But for this particular model, the optimal portfolio choice is a nonlinear function of the bifurcation portfolio, which leads to global inaccuracy. Second, one may be surprised to see that our linear approximation is not exactly tangent to the curve it is approximating. This results from the fact that we have computed two terms for the Taylor expansion of the *bifurcation portfolio* not of the portfolio itself. To recover the original curve, one would need to apply the second part of the bifurcation theorem, implying more involved calculations as noted in section 3.

Is the proposed approximation accurate enough ? Staying in the space of portfolio decision rules, we observe allocation errors ranging from 2% to 9% at the various approximation points. Whether such results are accurate enough to comment is a matter of personal appreciation. But it is clear that the validity of approximation decreases when wealth fluctuates away from the approximation point. This is a problem since the ergodic distribution of wealth is very wide in this kind of model with a close-to-unit-root behavior.

On the other hand, one has to keep in mind that portfolio choice is a second order optimization in that model, meaning that welfare gains produced by optimizing portfolios are likely to be small when compared to the gains brought by inter-temporal smoothing of consumption. It follows that if the accuracy measure is associated to welfare gains, the errors made on portfolio choice will be small when normalized by the gains of going from first order to second order approximations for the rest of the model. This is exactly what we see in figure

4.2 if we look at the various approximation points. The magnitude of the gains made on the savings equation by increasing the approximation order are higher than the gains made on the optimal allocation equation.

5 Example 2: two endowment economies

This section presents a two-countries example. Except for the notation, this model is very close to the one presented presented in Judd et al. [2002]. However, because we want to assess the precision of an essentially local method, we don't include any short-sale or borrowing constraint. It is thus a model with local incompleteness by analogy to the concept of local completeness used in Coeurdacier et al. [2010]. The same setup has been used in Devereux and Sutherland [2009a,b] to demonstrate the feasibility of their portfolio solution method.

The world consists in two countries indexed by $i = 1, 2$. They enjoy two sources of revenues: undiversifiable labor income $\left(w_t^i\right)$ and capital income $\left(d_t^i\right)$. In order to limit the number of state variables we assume that these exogenous process are independently distributed. We denote by $\left(y_t^i\right)$ the total revenue $\left(w_t^i + d_t^i\right)$.

Both countries can trade at price p_t^f a short-lived risk-free bond yielding b_t. They can also trade at price $\left(p_t^i\right)$ two short short-lived equity claims yielding $\left(x_t^i d_{t+1}^i\right)$ in next period period $t+1$. Available income W_t^i is a function of commitments made the period before:

$$\begin{aligned} W_t^1 &= y_t^1 + b_{t-1} - x_{t-1}^1 d_t^1 + x_{t-1}^2 d_t^2 \\ W_t^2 &= y_t^2 - b_{t-1} + x_{t-1}^1 d_t^1 - x_{t-1}^2 d_t^2 \end{aligned} \tag{5.1}$$

After trading in financial markets, consumption c_t^i enjoyed by country i is:

$$\begin{aligned} c_t^1 &= W_t^1 - b_t p_t + x_t^1 p_t^1 - x_t^2 p_t^2 \\ c_t^2 &= W_t^2 + b_t p_t - x_t^1 p_t^1 + x_t^2 p_t^2 \end{aligned} \tag{5.2}$$

I assume that consumption is valued using an inter-temporal utility function with C.R.R.A. instantaneous felicity. Each country maximizes $\sum_{t\geq 0} \beta^t \frac{(c_t)^{1-\gamma}}{1-\gamma}$ with $\beta \in [0, 1[$ and $\gamma > 1$.

Using the Euler equation associated with bond choice for each agent, I write a bond pricing function:

$$\frac{1}{2}\beta E_t \left[\left(\frac{c_{t+1}^1}{c_t^1} \right)^{-\gamma} + \left(\frac{c_{t+1}^2}{c_t^2} \right)^{-\gamma} \right] = p_t \tag{5.3}$$

and a market agreement equation:

$$\frac{1}{2}\beta E_t\left[\left(\frac{c^1_{t+1}}{c^1_t}\right)^{-\gamma}-\left(\frac{c^2_{t+1}}{c^2_t}\right)^{-\gamma}\right]=0 \tag{5.4}$$

Arbitrage equations for stocks are obtained in a similar way. Equities are priced using:

$$\begin{aligned}
\frac{1}{2}\beta E_t\left[\left\{\left(\frac{c^1_{t+1}}{c^1_t}\right)^{-\gamma}+\left(\frac{c^2_{t+1}}{c^2_t}\right)^{-\gamma}\right\}d^1_{t+1}\right] &= p^1_t \\
\frac{1}{2}\beta E_t\left[\left\{\left(\frac{c^1_{t+1}}{c^1_t}\right)^{-\gamma}+\left(\frac{c^2_{t+1}}{c^2_t}\right)^{-\gamma}\right\}d^2_{t+1}\right] &= p^2_t
\end{aligned} \tag{5.5}$$

while optimal allocations defined by:

$$\begin{aligned}
\frac{1}{2}\beta E_t\left[\left\{\left(\frac{c^1_{t+1}}{c^1_t}\right)^{-\gamma}-\left(\frac{c^2_{t+1}}{c^2_t}\right)^{-\gamma}\right\}d^1_{t+1}\right] &= 0 \\
\frac{1}{2}\beta E_t\left[\left\{\left(\frac{c^1_{t+1}}{c^1_t}\right)^{-\gamma}-\left(\frac{c^2_{t+1}}{c^2_t}\right)^{-\gamma}\right\}d^2_{t+1}\right] &= 0
\end{aligned} \tag{5.6}$$

Note that by contrast to Devereux and Sutherland [2009a], Tille et al. [2010] there is no need to write the portfolio equations 5.6 in any specific way. In particular it is not necessary that one or both of the factors appearing in the expectation operator be zero in the steady-state.

In this simple setting, there are 2 state variables W^1_t, W^2_t whose evolution is given in equation 5.1 and 6 control variables $p^f_t, b_t p^1_t, p^2_t, x^1_t, x^2_t$ associated to equations 5.3,5.4,5.5,5.6. There are also two auxiliary variables c^1_t, c^2_t depending on contemporaneous variables that can be substituted in other equations according to definition 5.2.

In order to study the effect of increased market incompleteness discussed in chapter **??**, we can shut down the market for equities by replacing equations 5.6 by:

$$\begin{aligned}
x^1_t &= 0 \\
x^2_t &= 0
\end{aligned}$$

5.1 Perturbation solution and bifurcation portfolios

Under perfect foresight the allocation between bonds and stocks is degenerate because the three allocation equations 5.4 and 5.6 are all equivalent to $\left(\frac{c^1_{t+1}}{c^1_t}\right) = \left(\frac{c^2_{t+1}}{c^2_t}\right)$. Using the pricing equations, we get that all assets yield the same returns: $\frac{1}{p_t} = \frac{d^1_{t+1}}{p^1_t} = \frac{d^2_{t+1}}{p^2_t}$.

The formulation of the model, naturally suggests to take x_t^1 and x_t^2 as portfolio variables with 5.6 as the associated portfolio criterion. However, we still need to reformulate the model so that under perfect foresight the solution of the non-portfolio equations doesn't depend on the portfolio choice. Hence we replace the amount of debt b_t (which depends on the choice of x_t^1, x_t^2), by the net-foreign asset position $N_t = b_t p_t - x_t^1 p_t^1 + x_t^2 p_t^2$ and rewrite the transition and the budget equation as:

$$\begin{aligned} W_t^1 &= y_t^1 + N_{t-1}\frac{1}{p_{t-1}^f} - x_{t-1}^1\left(d_t^1 - \frac{p_{t-1}^1}{p_{t-1}^f}\right) + x_{t-1}^2\left(d_t^2 - \frac{p_{t-1}^2}{p_{t-1}^f}\right) \\ W_t^2 &= y_t^2 - N_{t-1}\frac{1}{p_{t-1}^f} + x_{t-1}^1\left(d_t^1 - \frac{p_{t-1}^1}{p_{t-1}^f}\right) - x_{t-1}^2\left(d_t^2 - \frac{p_{t-1}^2}{p_{t-1}^f}\right) \end{aligned} \tag{5.7}$$

$$\begin{aligned} c_t^1 &= W_t^1 - N_t \\ c_t^2 &= W_t^2 + N_t \end{aligned} \tag{5.8}$$

While strictly equivalent to former equations 5.1 and 5.2, these new equations are expressed in terms of excess payoffs over the risky-free asset. This is the same manipulation as described in Devereux and Sutherland [2009a].[3]

We apply the procedure from section 3 at the symmetric deterministic steady-state $\bar{s} = \left(\overline{W^1}, \overline{W^2}\right) = \left(\overline{y^1}, \overline{y^2}\right)$. It yields second-order decision rules (Taylor expansions at $\bar{s}$) for the non-portfolio controls p_t^f, b_t, p_t^1, p_t^2 and first order solutions for x_t^1, x_t^2 (the first order coefficient of the bifurcation portfolio).

5.2 Calibration

Parameters values are summarized in table 4. The total output is normalized to 1 in each country with a 50% share on which contingent claims can be issued.

If I choose very small shocks we know that the bifurcation method will be asymptotically exact and that the true solution will feature a unit root because there will be no precautionary behavior anymore. Hence I consider standard deviations for the random innovations that are on the upper side equal to 5% in the baseline (see chapter **??** for a discussion on plausible calibrations). We also consider an asymmetric case where country 1 bears a lower risk than country 2 with the standard deviations of the shocks equal to 4%.

I assume there is no correlation between diversifiable and non-diversifiable incomes, but assume a cross-country correlation $\zeta = 0.5$. according to chapter

[3] Each term $x_{t-1}^i\left(d_t^i - \frac{p_{t-1}^i}{p_{t-1}^f}\right)$ could also be written as $x_{t-1}^1 p_{t-1}^1\left(\frac{d_t^1}{p_{t-1}^1} - \frac{1}{p_{t-1}^f}\right) = \alpha_{t-1}^1\left(r_t^1 - r_t^f\right)$ where α_t^i is the value if investment in stock i and $r_t^i - r_t^f$ the realization of excess returns against the risk-free asset. We have denoted net foreign asset position by N_t instead of W_t in order to avoid notation conflict with available income W_t^i.

	baseline	asymmetric
β	0.96	0.96
γ	4	4
$\bar{d}^1 = \bar{d}^2$	0.5	0.5
$\bar{w}^1 = \bar{w}^2$	0.5	0.5
σ_{ϵ_1}	0.05	0.04
σ_{ϵ_2}	0.05	0.05
σ_{η_1}	0.05	0.04
σ_{η_2}	0.05	0.05
ζ	0.5	0.5

Table 4: Calibration

?? this is necessary to increase the amount of aggregate risk needed to induce a stable behavior. I also exclude autocorrelation of the shocks in order to keep the state-space small.

5.3 Qualitative comparison with other solution methods

Given the choice of a small state-space, I can easily compare the bifurcation portfolios with a global collocation method. The rather standard time-iteration algorithm I use is sketched in appendix **??**. I consider two variants using either cubic splines or Smolyak products of Chebychev polynomials. In that particular instance, they yields very similar results. I also include a comparison with the risky steady-state solution proposed in chapter **??**. [4]

Figures 5.1 show the decision rules for the bond-only economy. The plots are made against relative wealth $\frac{W_{1,t}-W_{2,t}}{2} = b_t$ (the first line) and aggregate wealth $\frac{W_{1,t}-W_{2,t}}{2} = \epsilon_t$ (second line). The representation of Smolyak collocation and splines decision rules are almost identical. They depart from a standard linear approximation by the following qualitative features.

First, the price of the risk-less bond features is adjusted by the expected variance of consumption. With some overshooting, this is reflected in the second order and risky steady-state solutions which both include a precautionary term in the expansion of the Euler equation (upper left part). In addition to that, the second-order solution captures well the nonlinearity in the price sensitivity to aggregate wealth (lower left part).

Second, the global solution features a locally stable behavior: starting from an initially non zero level of debt b_t and if there is no shock, the level of debt next period is b_{t+1} smaller than b_t in absolute terms. It is informative to consider the biggest eigenvalue of the transition function at the steady-state. These eigenvalues are reported in table 5. In contrast to standard perturbation methods which have a dominant eigenvalue equal to 1, the two global methods have

[4] The comparisons with the risky steady-state and considerations associated with the bond-only economy are only meant for the present dissertation and absent from the working-paper version.

Biggest eigenvalue	Symmetric case		Asymmetric case	
	Bonds only	Bonds and stocks	Bonds only	Bonds and stocks
Perturbations (1st order)	1.	1.	1.	1
Perturbations (2nd order)	1.	0.998	1.	0.993
Linear risky steady-state	0.784	0.998	0.803	0.992
Smolyak (d=4)	0.926	0.967	0.945	0.973
Splines	0.929	0.967	0.939	0.973

Table 5: Biggest eigenvalues

Risky steady-states		W_1	W_2
Bonds only			
	Linear risky steady-state	0.963	1.037
	Smolyak (d=4)	0.951	1.049
	Splines	0.955	1.045
Bonds and stocks			
	Linear risky steady-state	0.963	1.037
	Smolyak (d=4)	0.940	1.060
	Splines	0.940	1.060

Table 6: Risky steady-states asymmetric case

biggest eigenvalues equal to 0.926 and 0.0929 respectively. These values are significantly smaller than 1: the associated half life is about 9 periods. This confirms the existence of a locally stable behavior due to precautionary behavior, as described in **??**. While we have provided some intuition for this behavior using a linear approximation around the risky steady-state, it appears that numerically this approximation greatly exaggerates the convergent behavior by producing 0.784 as the biggest eigenvalue, corresponding to a half-life of roughly 3 years. Stationarity can also be assessed graphically: figure 5.4 shows the ergodic distribution approximated by the density of states after 40 periods, using 10000 random draws.

With asymmetric shocks, the risky steady-state is not necessarily equal to 0. Intuitively, when one country is riskier than another, in equilibrium it must compensate a higher level of risk by a higher level of consumption. Figure 5.2 and table 6 show the risky steady-states associated to the various approximation functions.

Note that because the collocation methods approximates the decision rule on a finite state-space, I extrapolate the decision rule linearly for points that are slightly outside. There is no precautionary savings due to occasionally binding constraints, but there are some numerical glitches caused by the extrapolation close to the boundaries. For instance in figure 5.1, there is a spurious dependence on the price on the net foreign position appearing close to the bounds. We check that it doesn't affect our results by changing the boundaries of the approximation space and checking that local behavior around the steady-state is unchanged.

After introducing equity trading in the economy, we obtain eigenvalues that are closer to 1 (see table 5). We can verify that completing the markets leads to more persistent behavior. It reads as follows: in our model, the aggregate wealth is exogenous and hence clearly mean reverting. So, persistent behavior is necessarily associated to persistence in relative wealth. With complete markets, agents would insure one another so that their incomes would be perfectly correlated and there would be no fluctuation at all in relative wealth. In other terms any initial deviation in relative wealth would persist forever. Hence we have a monotonic dependence of the biggest eigenvalue with respect to market completeness: it is 0 in autarky, 0.92 with bonds, 0.97 with bonds and stocks, and 1 under complete markets. Intuitively, the higher the market incompleteness, the higher the need to self-insure against non tradable shocks. Interestingly, this intuition extends to the case of borrowing constraints: if agents are prevented to exchange in certain states of the world (here were debt is too high), they would try to avoid it by staying away from the constraints. Anagnostopoulos [2004] shows exactly this effect in a setup similar to ours.

Under complete markets, the situation is obscured by the fact that, conditionally on the initial wealth, the state-space in which the economy fluctuates can be reduced by one dimension, usually producing a stable evolution law for the remaining states. When markets are incomplete this is not an option, and because usual perturbation notably suffer from a unit-root problem in this case, it may induce the false belief that market incompleteness creates stationarity problem. I argue that it is merely an artifact of these perturbation methods, and that the exact opposite is theoretically possible.

Looking at the portfolio decision rules (table 5.3) we see that the first order bifurcation portfolio captures the portfolio choice relatively well. In the symmetric equilibrium each country sells half of its own stocks to get the same exposure to local and foreign shocks. According to the bifurcation solution, when a country gets richer by an amount 0.2, it borrows 0.2 and buys 0.2 worth of stock in each country to convert all its supplemental income in available assets. Using global projections we get almost the same portfolio composition although not all income is reinvested which makes the figures smaller. By contrast, the linear approximation around the risky steady-state produces a false prediction for bond holdings in the same scenario (it increases instead of decreasing).

5.4 Accuracy measures

In the preceding paragraph I have directly compared the bifurcation portfolios to other solution methods. Another approach to accuracy checks consists in defining accuracy measures that use the model equations directly to assess how close the solution is from solving the model.

All accuracy measures defined in appendix **??** use the residuals of Euler equations computed at each point of a fine regularly spaced grid. They differ from each other by the weights associated to each state.

I find that second-order approximation and global methods reduce greatly

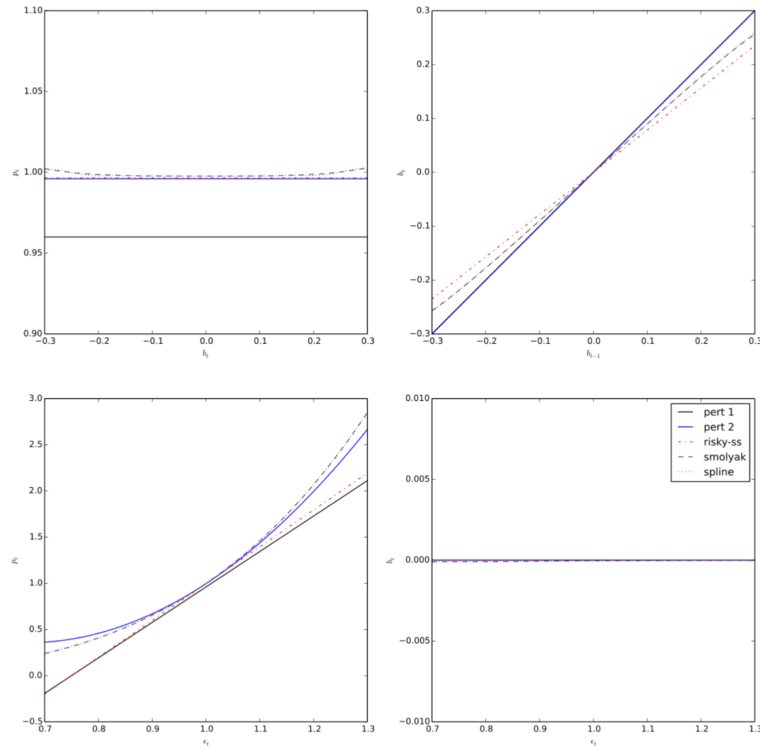

Figure 5.1: Decision rules with bonds only

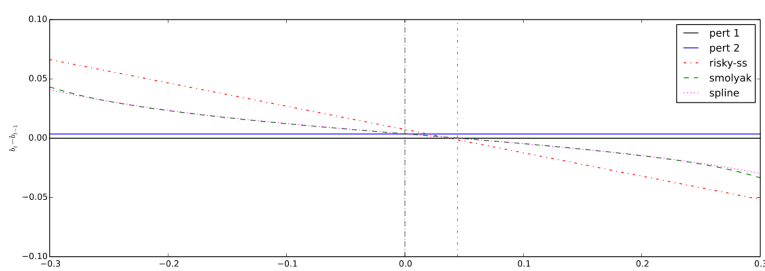

Figure 5.2: Bonds accumulation (asymmetric case)

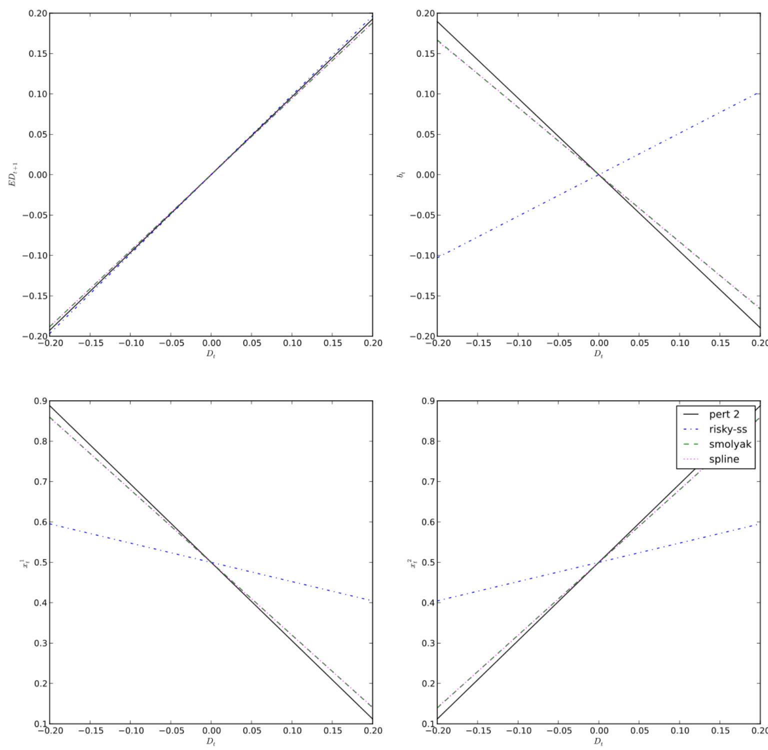

Figure 5.3: Decision rules for portfolios

Accuracy measures	$\Omega_\infty(\varphi)$	$\Omega_1(\varphi)$	$\Omega_{1,\infty}(\varphi)$	$\Omega_{1,\infty,\widetilde{\varphi}}(\varphi)$	$\Omega(\bar{s},\varphi)$
Bonds only					
Perturbations (1st order)	2.0100	0.4354	0.2901	0.2262	0.1184
Perturbations (2nd order)	0.9943	0.1141	0.0417	0.0337	0.0200
Linear risky steady-state	1.6289	0.3319	0.0939	0.0874	0.0698
Smolyak (d=4)	0.0486	0.0156	0.0141	0.0140	0.0136
Splines	0.0408	0.0147	0.0136	0.0136	0.0135
Bonds and stocks					
Perturbations (2nd order)	0.425	0.079	0.057	0.058	0.058
Linear risky steady-state	8.767	0.271	0.079	0.096	0.104
Smolyak (d=4)	0.046	0.017	0.016	0.016	0.016
Splines	0.046	0.017	0.016	0.016	0.016

Table 7: Error measures

the worse error on the grid (measure Ω_∞). Graph 7 shows that errors of first order decision rule are clearly anisotropic. They depend on the level of aggregate income and much less on relative wealth. A glance at the decision rule (figure 5.1) suggests that this is due to the non linearity of the price with respect to aggregate wealth (lower left part). It also explains why the second-order solution performs much better than the first order one.

Weighting the errors by the ergodic distribution shown in figure 5.4 to get $\Omega_{1,\infty}$ improves the performance of second order approximations: it is now 7.5 times better than first order perturbations (instead of 2 times) and 3 times worse than global projections (instead of 40 times). This result is virtually unmodified if we use the distribution generated by the more precise method (splines) instead of a different one for each method (measure $\Omega_{1,\infty,\widetilde{\varphi}}$).

All these conclusions stand when we consider the asymmetric setup, or the economy with bonds and stocks. However, in this last case the relevance of error measures is questionable as I have already pointed out in section 4. Given that portfolio choices produce second order welfare gains, errors made by mis-allocating portfolios may be negligible compared to errors in the net foreign asset position or in bond pricing.

As for the linear approximation around the risky steady-state, the worst-case measure shows that it always perform better than usual linear approximation. The weighted measures also present this method favorably: its precision is about 2.5 times better than first order perturbations and about 2 times worse than second order-perturbations. This is especially surprising when we compare the graphs 5.5 and 5.7: in the asymmetric case, the accuracy of the risky steady-sate approximation is actually lower where the economy spends more time, i.e. for positive bond holdings. Again, this shows that synthetic accuracy measures may potentially produce spurious rankings of solution methods.

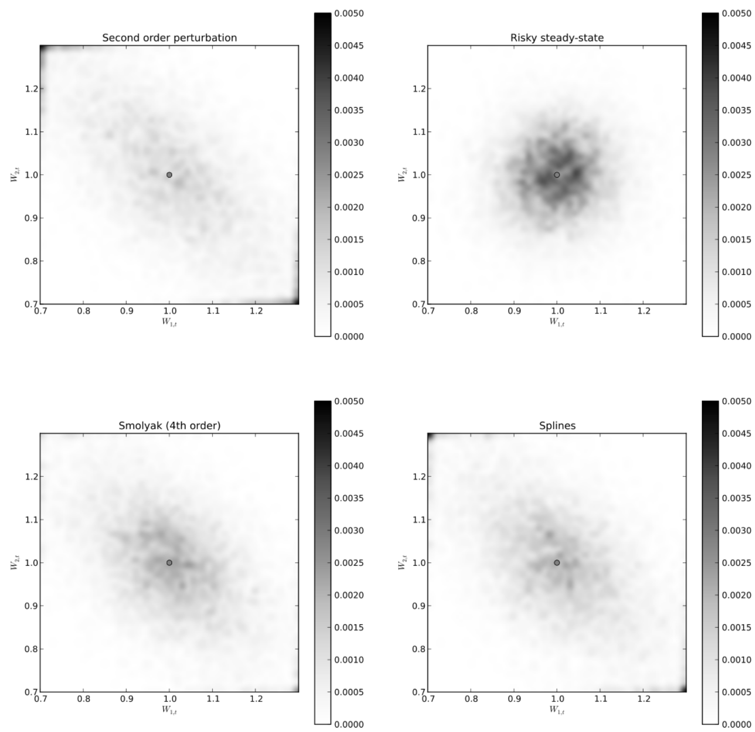

Figure 5.4: Ergodic states

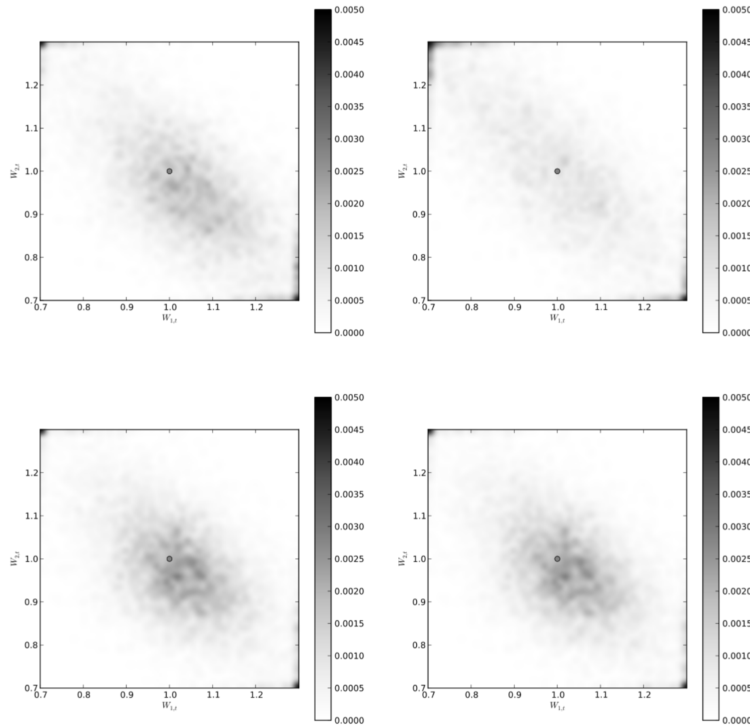

Figure 5.5: Ergodic states (asymmetric case)

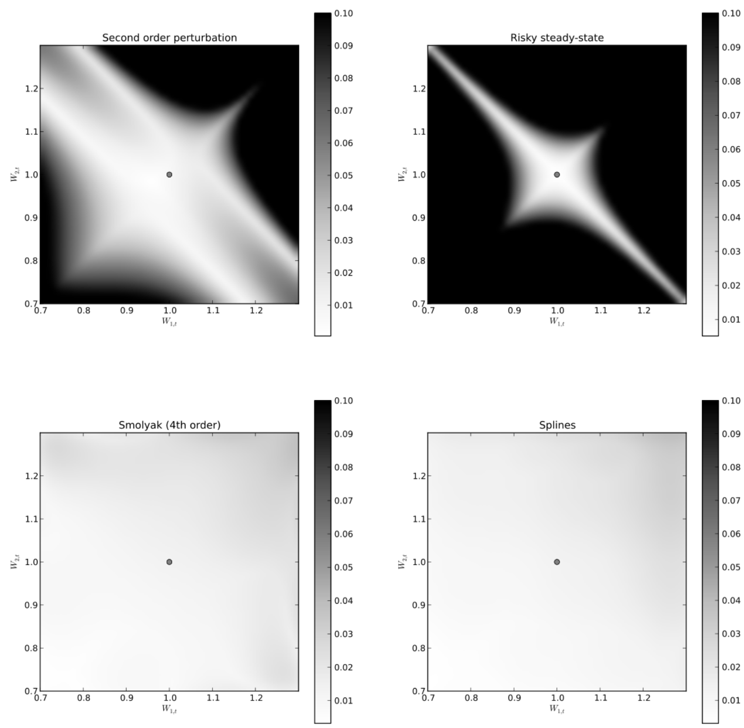

Figure 5.6: Errors - bonds - symmetric case

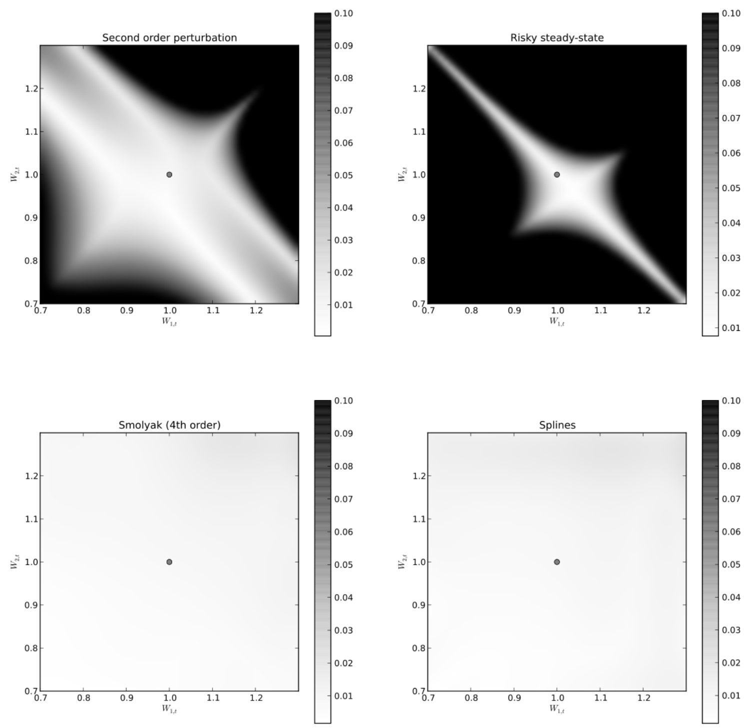

Figure 5.7: Errors - bonds - asymmetric case

Accuracy measures (asymmetric case)	$\Omega_\infty(\varphi)$	$\Omega_1(\varphi)$	$\Omega_{1,\infty}(\varphi)$	$\Omega_{1,\infty,\widetilde{\varphi}}(\varphi)$	$\Omega(\bar{s},\varphi)$
Bonds only					
Perturbations (1st order)	2.0100	0.4354	0.2901	0.2262	0.1184
Perturbations (2nd order)	0.9943	0.1141	0.0417	0.0337	0.0200
Linear risky steady-state	1.6289	0.3319	0.0939	0.0874	0.0698
Smolyak (d=4)	0.0486	0.0156	0.0141	0.0140	0.0136
Splines	0.0408	0.0147	0.0136	0.0136	0.0135
Bonds and stocks					
Perturbations (1st order)	4 10^8	3 10^5	0.0471	0.0531	0.0550
Perturbations (2nd order)	0.3837	0.0553	0.0198	0.0204	0.0208
Linear risky steady-state	3.549	0.2192	0.0396	0.0497	0.0549
Smolyak (d=4)	0.0533	0.0205	0.0188	0.0191	0.0191
Splines	0.0533	0.0204	0.0188	0.0191	0.0191

Table 8: Error measures (asymmetric case)

6 Conclusion

In this chapter, we have tried to clarify how the perturbation method can be extended to solve for dynamic portfolio choices using the bifurcation theory. It has connected the original results by Judd and Guu [2001] for a static portfolio choice, with the computations made by Devereux and Sutherland [2009a], Tille et al. [2010] in a dynamic two-countries model, and provided some guidance to assess the validity of other similar applications in a DSGE context. In practice this suggests that a very necessary criterion for the computation of bifurcation portfolios, is the requirement that a given portfolio term should not depend on the computation on a higher order term.

As for the numerical results my own interpretation is the following. First, it is well known that even static portfolios are quite instable with respect to the the parameters defining them. We have yet another manifestation of that effect here, where sizable errors on the portfolios can be compatible with small approximation errors of the model. In the perturbation case, we see that performing a second order approximation leads to higher accuracy gains than computing optimal portfolios.

Second, the partial equilibrium model shows portfolios featuring a strong nonlinearity with the level of wealth, hence deviating notably from the linear portfolios that I have produced. However, the open-economy application has portfolios that are very close to linear and are thus accurately approximated by the bifurcation ones. This is clearly a general equilibrium effect where the nonlinearity of both agent's preferences compensate each other almost exactly. Of course there is no reason for it to be true in any general equilibrium model. One immediate extension of this present work would consist in assessing how nonlinear portfolios are when preferences are asymmetric across countries, or when we consider other important modeling variables such as capital or multiple goods.

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